This is a digital copy of a book that was preserved for generations on library shelves before it was carefully scanned by Google as part of a project to make the world's books discoverable online.

It has survived long enough for the copyright to expire and the book to enter the public domain. A public domain book is one that was never subject to copyright or whose legal copyright term has expired. Whether a book is in the public domain may vary country to country. Public domain books are our gateways to the past, representing a wealth of history, culture and knowledge that's often difficult to discover.

Marks, notations and other marginalia present in the original volume will appear in this file - a reminder of this book's long journey from the publisher to a library and finally to you.

Usage guidelines

Google is proud to partner with libraries to digitize public domain materials and make them widely accessible. Public domain books belong to the public and we are merely their custodians. Nevertheless, this work is expensive, so in order to keep providing this resource, we have taken steps to prevent abuse by commercial parties, including placing technical restrictions on automated querying.

We also ask that you:

+ *Make non-commercial use of the files* We designed Google Book Search for use by individuals, and we request that you use these files for personal, non-commercial purposes.

+ *Refrain from automated querying* Do not send automated queries of any sort to Google's system: If you are conducting research on machine translation, optical character recognition or other areas where access to a large amount of text is helpful, please contact us. We encourage the use of public domain materials for these purposes and may be able to help.

+ *Maintain attribution* The Google "watermark" you see on each file is essential for informing people about this project and helping them find additional materials through Google Book Search. Please do not remove it.

+ *Keep it legal* Whatever your use, remember that you are responsible for ensuring that what you are doing is legal. Do not assume that just because we believe a book is in the public domain for users in the United States, that the work is also in the public domain for users in other countries. Whether a book is still in copyright varies from country to country, and we can't offer guidance on whether any specific use of any specific book is allowed. Please do not assume that a book's appearance in Google Book Search means it can be used in any manner anywhere in the world. Copyright infringement liability can be quite severe.

About Google Book Search

Google's mission is to organize the world's information and to make it universally accessible and useful. Google Book Search helps readers discover the world's books while helping authors and publishers reach new audiences. You can search through the full text of this book on the web at `http://books.google.com/`

Mathematics
QA
295
.M167s
B 503921

SOME THEOREMS IN THE THEORY OF SUMMABLE DIVERGENT SERIES

157

BY

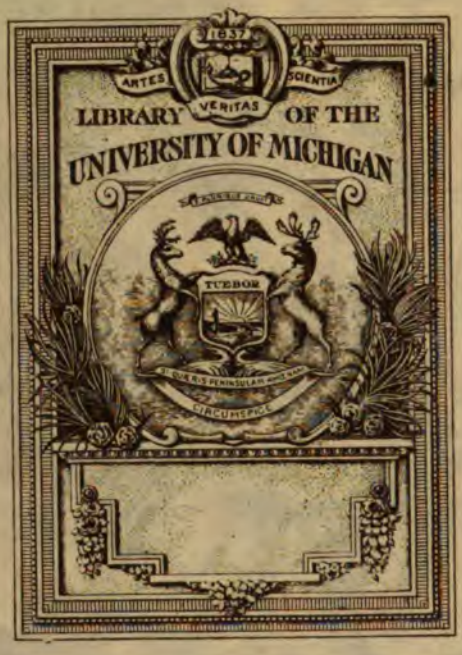
FRANK JOSEPH McMACKIN

DISSERTATION

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY, IN THE FACULTY OF PURE SCIENCE, COLUMBIA UNIVERSITY

PRESS OF
THE NEW ERA PRINTING COMPANY
LANCASTER, PA.
1918

Digitized by Google



Digitized by Google

SOME THEOREMS IN THE THEORY OF SUMMABLE DIVERGENT SERIES

BY

FRANK JOSEPH McMACKIN

DISSERTATION

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY, IN THE FACULTY OF PURE SCIENCE, COLUMBIA UNIVERSITY

PRESS OF
THE NEW ERA PRINTING COMPANY
LANCASTER, PA.
1916

Digitized by Google

Digitized by Google

INTRODUCTION

The present paper embodies the results of an investigation concerning the continuity of the sum function of summable divergent series. In particular the following problem is solved: To determine the necessary and sufficient conditions that a summable divergent series of functions with continuous terms must fulfill in order that its sum function, as defined by Borel, Cesàro, and Hardy and Chapman, be itself continuous.

Hardy* and Smail† both defined a summable series to be uniformly summable if the limit involved in the defined sum was approached uniformly. My first problem consists in determining whether this type of convergence is both necessary and sufficient for the continuity of the sum function of a series of continuous functions. This question is settled by the citation of a specific series whose Cesàro sum function is continuous but which is not uniformly summable.

Then in a manner somewhat analogous to the definition given for quasi-uniform convergence by Arzela,‡ I define what I call quasi-uniform summability of a summable divergent series. I then show that this type of summability is the necessary and sufficient condition for the continuity of the sum function of series summable by any of the three methods mentioned above. I do not attempt the investigation of this question in connection with the most general sums given for divergent series, as for example those given by Smail,§ but confine my attention to those sums which are most practicable and most frequently used in calculations.

Finally I show that if a series is quasi-uniformly convergent it is also quasi-uniformly summable, so that my definition of quasi-uniform summability is consistent with that of quasi-uniform convergence.

---

* G. H. Hardy: "On Diff. and Integration of Divergent Series," *Trans. Cambridge Philosophic Soc.*, Vol. 19, pp. 297–321.

† Lloyd L. Smail, Columbia University Dissertation, 1913.

‡ Sulla serie di funzioni, part 1, Bologna, 1899, p. 10 et seq. [See also Townsend, E. J., *Bull. Amer. Math. Soc.*, Vol. 12, p. 7.]

§ Loc. cit. Also a paper presented to the Amer. Math. Soc. (see *Bull. Amer. Math. Soc.*, Vol. 20, July, 1914), a copy of which the author very kindly sent me.

Digitized by Google

§ 1. AN EXAMPLE OF A NON-CONVERGENT SUMMABLE SERIES OF CONTINUOUS FUNCTIONS WHICH HAS A CONTINUOUS CESÀRO SUM FUNCTION, YET WHICH IS NOT UNIFORMLY SUMMABLE, ACCORDING TO HARDY

Consider the series such that

$$(1) \qquad s_n = -n \cdot (n-1)xe^{-(n-1)x^2} + (n+1)nx \cdot e^{-nx^2} + (-1)^n,$$

where s_n is the sum of the first $n+1$ terms of that series.

(*a*) This series is non-convergent—for any finite value of x—for

$$\lim_{n=\infty} s_n = \lim_{n=\infty} [-n \cdot (n-1)xe^{-(n-1)x^2} + (n+1)nxe^{-nx^2} + (-1)^n]$$
$$= 0 + \lim_{n=\infty} (-1)^n.$$

But $(-1)^n$ approaches no limit as n increases, but oscillates between 0 and $+1$.

(*b*) This series is summable by Cesàro's sum for $k = 1$. For

$$(2) \qquad \frac{s_0 + s_1 + \cdots + s_n}{n+1} = \frac{(n+1)nxe^{-nx^2} + [+1 - 1 + 1 \cdots + (-1)^n]}{n+1},$$

and

$$(3) \qquad S(x) = \lim_{n=\infty} \frac{s_0 + s_1 + \cdots + s_n}{n+1} = \lim_{n=\infty} nxe^{-nx^2} + 0 = 0,$$

for any finite value of x.

(*c*) The sum function $S(x)$ is continuous yet the series is not uniformly summable.

A series summable by Cesàro's sum is said to be uniformly summable provided the limit in (3) is approached uniformly, i. e., provided

$$(4) \qquad |nxe^{-nx^2}| < \epsilon,$$

$n > N$ and any x in some chosen interval say $(-a, a)$, where a is a finite number greater than zero. The inequality (4) does not hold, however, for x near zero,* and the series is not uniformly summable. It has thus been shown that uniform summability, while sufficient for the continuity of the sum function of a series of continuous functions, is not necessary.

§ 2. QUASI-UNIFORM APPROACH TO A LIMIT

We proceed now to determine some necessary and sufficient conditions under which the sum function of a summable series of continuous functions is itself continuous. To that end we define quasi-uniform approach to a limit.

Note.—The definition here given is more general than is needed for the subsequent discussion, but the general form is given because of its suggestiveness.

* Osgood, *Bull. Am. Math. Soc.*, Vol. 3.

Digitized by Google

Let there be given a function

$$f[x_1^{(1)}, x_2^{(1)}, \cdots, x_n^{(1)}, x_1^{(2)}, x_2^{(2)}, \cdots, x_m^{(2)}, \cdots, x_1^{(p)}, x_2^{(p)}, \cdots, x_q^{(p)}]$$

of p sets of variables $x_1^{(1)}, x_2^{(1)}, \cdots, x_n^{(1)}, x_1^{(2)}, x_2^{(2)}, \cdots, x_m^{(2)}, \cdots, \cdots, x_1^{(p)}, x_2^{(p)}, \cdots, x_q^{(p)}$, wherein for convenience we will represent the set $x_1^{(1)}, x_2^{(1)}, \cdots, x_n^{(1)}$ by $x^{(1)}$, $x_1^{(2)}, \cdots, x_m^{(2)}$ by $x^{(2)}$, etc. Furthermore when $x_1^{(1)} \to a_1^{(1)}, x_2^{(1)} \to a_2^{(1)}, \cdots, x_p^{(1)} \to a_n^{(1)}$, where the $a_i^{(1)}$ are some constants, we shall say simply $x^{(1)} \to a^{(1)}$.

Let the function f be defined for $x^{(1)}$ in an n-way space $X^{(1)}$, $x^{(2)}$ in an m-way space $X^{(2)}$, etc.

For each $x^{(1)}$ in $X^{(1)}$ let

$$\lim_{\substack{x^{(2)} \to a^{(2)} \\ \cdots \\ x^{(p)} \to a^{(p)}}} f[x^{(1)}, x^{(2)}, \cdots, x^{(p)}] = \varphi(x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, \cdots, x_n^{(1)}).$$

Let the numbers ϵ and $\delta^{(2)}, \delta^{(3)}, \cdots, \delta^{(p)}$ be arbitrarily chosen but all greater than zero. Then if there exists a set of numbers $D^{(2)}, D^{(3)}, \cdots, D^{(p)}$ all greater than zero and fixed, such that $\delta^{(2)} > D^{(2)}, \delta^{(3)} > D^{(3)}, \cdots, \delta^{(p)} > D^{(p)}$, and such that for each $x^{(1)}$ in $X^{(1)}$ there is one or several sets of numbers,

$$\delta_i^{(2)}, \delta_j^{(3)}, \cdots, \delta_l^{(p)},$$

where each $\delta_\alpha^{(\lambda)}$ is one of a finite set of numbers such that $\delta^{(\lambda)} > \delta_\alpha^{(\lambda)} > D^{(\lambda)}$, which make

$$|f(x^{(1)}, x^{(2)}, \cdots, x^{(p)}) - \varphi(x_1^{(1)}, x_2^{(1)}, \cdots, x_n^{(p)})| < \epsilon,$$

when

$$|x^{(2)} - a^{(2)}| = \delta_i^{(2)}, \quad |x^{(3)} - a^{(3)}| = \delta_j^{(3)}, \quad \cdots, \quad |x^{(p)} - a^{(p)}| = \delta_l^{(p)},$$

f is said to approach φ quasi-uniformly in $X^{(1)}$.

The above definition is constructed on the assumption that $x^{(2)} \to a^{(2)}$, $x^{(3)} \to a^{(3)}, \cdots, x^{(p)} \to a^{(p)}$ simultaneously but independently. It may, however, not be desirable in some work to have all these limits approached simultaneously but rather according to some law of succession as in the case of repeated limits

$$\lim_{x^{(2)} \to a^{(2)}} \lim_{x^{(3)} \to a^{(3)}} \cdots \lim_{x^{(p)} \to a^{(p)}} f(x^{(1)}, x^{(2)}, \cdots x^{(p)}) = \varphi(x^{(1)})$$

or even in groups as

$$\lim_{\substack{x^{(2)} \to a^{(2)} \\ x^{(3)} \to a^{(3)} \\ \cdots \\ x^{(i)} \to a^{(i)}}} \cdots \lim_{\substack{x^{(i)} \to a^{(i)} \\ \cdots \\ x^{(p)} \to a^{(p)}}} f(x^{(1)}, x^{(2)}, \cdots, x^{(p)}) = \varphi(x^{(1)}).$$

Digitized by Google

In such cases f might be said to approach φ through repeated limits and the method of approach definitely stated.

As an example of the above definition consider an infinite series of functions $\Sigma a_n(x)$, the f of the definition is in this case $S_n(x)$, a function of two sets of variables each set containing a single member x and n. f is defined in a one-way space through integral values from 0 to ∞, while x is defined in a one-way space through all real values from a to b.

Set $n = 1/p$ then $S_n(x)$ becomes a function of x and p which we may write $S(x, p)$. Then assuming that the series is defined for each x in (a, b) we would have

$$\lim_{n\to\infty} S_n(x) = \lim_{p\to 0} S(x, p) = s(x).$$

Then, if to ϵ and $\delta^{(p)}$ arbitrarily chosen but greater than 0, as $\delta^{(p)} = 1/N$, where N is an arbitrarily large integer, there corresponds a $D^{(p)} > 0$ such that $\delta^{(p)} > D^{(p)} > 0$, such that for each x in (a, b)

$$|f(x, p) - \varphi(x)| < \epsilon,$$

that is

$$|S(x, p) - s(x)| < \epsilon,$$

when $|p - 0| = \delta_i^{(p)}$, where $\delta_i^{(p)}$ is one of a finite set such that $\delta^{(p)} > \delta_i^{(p)} > D^{(p)}$, then $S(x, p)$ approaches $s(x)$ quasi-uniformly in (a, b). We note that this is precisely the definition for quasi-uniform convergence given by Arzela.*

Again in the case of a Pringsheim double limit

$$\lim_{p, q\to\infty} S_{p, q}(x) = s(x) \qquad x \text{ in } (a, b),$$

we have three sets of variables, each set containing a single member x, p and q; or they might be considered as two sets, one containing the single member x, the other the two members p and q.

Set $p' = 1/p$, $q' = 1/q$, then in either of the above cases we should say that if to ϵ and $\delta^{(2)}$ and $\delta^{(3)}$ arbitrarily chosen but greater than zero there correspond two numbers $D^{(2)}$ and $D^{(3)}$ both greater than zero and such that

$$\delta^{(2)} > D^{(2)}, \qquad \delta^{(3)} > D^{(3)},$$

and such that, further, for each x in (a, b)

$$\left| S_{\frac{1}{p'}, \frac{1}{q'}}(x) - s(x) \right| < \epsilon,$$

when $|p' - 0| = \delta_i^{(2)}$, $|q' - 0| = \delta_j^{(3)}$ where $\delta_i^{(2)}$ and $\delta_j^{(3)}$ are each one of a finite set satisfying the inequalities

$$\delta^{(2)} > \delta_i^{(2)} > D^{(2)}, \qquad \delta^{(3)} > \delta_j^{(3)} > D^{(3)},$$

the Pringsheim double limit is approached quasi-uniformly in (a, b).

* Loc. cit.

Digitized by Google

§ 3. THE BOREL INTEGRAL SUM

We proceed now to apply the above definition, or rather a special case of it, to the investigation of the problem suggested by the series given above, that is

What are the necessary and sufficient conditions that the sum function of an oscillating series of continuous functions be itself continuous?

The observation might be made at this point to the effect that any set of conditions that we might obtain for oscillating series must apply equally well to convergent series since the latter may be considered as having zero oscillations.

Since the Borel Integral Sum* of an oscillating series was one of the first sums discovered, and furthermore since it is still one of the most powerful that is used in calculations we begin by considering that sum from the point of view of the question just put.

Definition.—A summable series of functions is said to be quasi-uniformly summable (B) in an interval (a, b), when the Borel integral is quasi-uniformly convergent in that interval.

By our definition of p. 3 the integral

$$\int_0^X f(\alpha, x)dx$$

defined for all values of α in (a, b), and for all positive values of X, will converge quasi-uniformly for α in (a, b) as X passes through all positive values, provided that to ϵ and N arbitrarily chosen there corresponds a fixed positive number $N' > N$ such that for each α in (a, b)

$$\left| \int_{n_\alpha}^{\infty} f(\alpha, x)dx \right| < \epsilon$$

for $N' > n_\alpha > N$, where n_α is one of a finite set of values between N and N'.

THEOREM I. *Let $f(\alpha, x)$ be a continuous function of x in the interval $(0, X)$, X an arbitrarily large positive number, and of α in the interval (a, b). Let further*

$$\int_0^{\infty} f(\alpha, x)dx$$

converge for every value of α in (a, b).

Then, the necessary and sufficient condition that the integral

$$J(\alpha) = \int_0^{\infty} f(\alpha, x)dx$$

* Borel, E., "Leçons sur les séries divergentes." Paris, 1901.

Digitized by Google

*be a continuous function of α in (a, b) is that it converge quasi-uniformly in (a, b).**

Proof.—(a) *The condition is sufficient.*

By simple convergence, for every α in (a, b), say $\alpha = \alpha'$, we have

$$\left|\int_n^\infty f(\alpha', x)dx\right| < \frac{\epsilon}{3} \qquad \text{for any } n > N. \tag{1}$$

Again from the quasi-uniform convergence of the integral, to $\epsilon/3$ and N arbitrarily chosen, there corresponds an $N' > N$ such that for every α in (a, b) there exists an n_α, one of a finite set lying between N and N', and such that

$$\left|\int_{n_\alpha}^\infty f(\alpha, x)dx\right| < \frac{\epsilon}{3}. \tag{2}$$

Furthermore from the continuity of $f(\alpha, x)$ we know also that the integral

$$\int_0^X f(\alpha, x)dx,$$

where X is any fixed positive number, is a continuous function of α in (a, b). So for a chosen X and $\epsilon/3$ we have

$$\left|\int_0^X f(\alpha', x)dx - \int_0^X f(\alpha'', x)dx\right| < \frac{\epsilon}{3}$$

provided $|\alpha' - \alpha''| < \delta_X$, where δ_X indicates that δ depends essentially on the X chosen.

If, now, X represent each of the finite number of n_α lying between N and N' above, there will correspond to these X's a finite number of δ_X from which we choose the smallest say δ. Then

$$\left|\int_0^{n_{\alpha''}} f(\alpha', x)dx - \int_0^{n_{\alpha''}} f(\alpha'', x)dx\right| < \frac{\epsilon}{3} \tag{3}$$

when $|\alpha' - \alpha''| < \delta$ and $n_{\alpha''}$ represents some one of the n_α's between N and N'.

From the inequalities (1) and (2) we may write

$$\left|\int_{n_{\alpha''}}^\infty f(\alpha'', x)dx\right| < \frac{\epsilon}{3} \quad \text{and} \quad \left|\int_{n_{\alpha''}}^\infty f(\alpha', x)dx\right| < \frac{\epsilon}{3},$$

the latter one since $n_{\alpha''} > N$.

Combining (3) with these we obtain

$$\left|\int_0^{n_{\alpha''}} f(\alpha', x)dx - \int_0^{n_{\alpha''}} f(\alpha'', x)dx\right| + \left|\int_{n_{\alpha''}}^\infty f(\alpha', x)dx - \int_{n_{\alpha''}}^\infty f(\alpha'', x)dx\right| < \epsilon$$

* Compare Bromwich, *London Math. Soc. Proc.*, Vol. 1, Ser. 2, p. 199. There is there given a set of conditions for the continuity of $J(\alpha)$, but his assumptions are somewhat different from the above and the conditions are not quite so neatly put, as those above.

Digitized by Google

for $|\alpha' - \alpha''| < \delta$. Or,

$$\left|\int_0^{n_{\alpha''}} f(\alpha', x)dx + \int_{n_{\alpha''}}^{\infty} f(\alpha', x)dx - \left[\int_0^{n_{\alpha''}} f(\alpha'', x)dx + \int_{n_{\alpha''}}^{\infty} f(\alpha'', x)dx\right]\right| < \epsilon.$$

Consequently

$$\left|\int_0^{\infty} f(\alpha', x)dx - \int_0^{\infty} f(\alpha'', x)dx\right| < \epsilon$$

for $|\alpha' - \alpha''| < \delta$, which establishes the continuity of $J(\alpha)$ in (a, b), and so part of our theorem is demonstrated.

(*b*) *The condition is necessary.*

The integral $J(\alpha)$ converges for every α in (a, b) and is continuous in (a, b). Also the integral

$$\int_0^x f(\alpha, x)dx$$

is continuous in (a, b).

Let α' be a point of (a, b), then to $\epsilon/3$ there corresponds a δ such that

$$\left|\int_0^{\infty} f(\alpha, x)dx - \int_0^{\infty} f(\alpha', x)dx\right| < \frac{\epsilon}{3}, \quad \text{for } |\alpha - \alpha'| < \delta_1. \tag{4}$$

Since the integral converges for every α in (a, b), to $\epsilon/3$ there corresponds an N such that

$$\left|\int_n^{\infty} f(\alpha', x)dx\right| < \frac{\epsilon}{3} \qquad \text{for any } n > N. \tag{5}$$

The number n being thus chosen, to $\epsilon/3$ and this n, there corresponds a δ_2 which gives

$$\left|\int_0^{n} f(\alpha, x)dx - \int_0^{n} f(\alpha', x)dx\right| < \frac{\epsilon}{3} \qquad \text{provided } |\alpha' - \alpha| < \delta_2. \tag{6}$$

Furthermore, the following inequality holds:

$$\begin{aligned}\left|\int_n^{\infty} f(\alpha, x)dx\right| < &\left|\int_0^{\infty} f(\alpha, x)dx - \int_0^{\infty} f(\alpha', x)dx\right| \\ &+ \left|\int_0^{n} f(\alpha, x)dx - \int_0^{n} f(\alpha', x)dx\right| + \left|\int_n^{\infty} f(\alpha', x)dx\right|.\end{aligned} \tag{7}$$

Choosing now the smaller of δ_1 and δ_2 from (4) and (6), say δ, and combining (4), (5) and (6) with (7) we obtain

$$\left|\int_n^{\infty} f(\alpha, x)dx\right| < \epsilon \qquad \text{for } |\alpha - \alpha'| < \delta,$$

n being some fixed number greater than N.

Since $J(\alpha)$ is continuous over the whole interval (a, b) a number $n > N$

Digitized by Google

may be determined for each point α of that interval, and also a partial interval having that point as mid-point, such that

$$\left| \int_n^\infty f(\alpha, x)dx \right| < \epsilon,$$

for any α in that partial interval.

Consequently all points of the interval (a, b) are interior to some such partial interval, so that by the Heine-Borel* theorem there exist a finite number of such intervals. So, to each α in (a, b) there can be made correspond a number n_α, of a finite set, such that

$$\left| \int_{n_\alpha}^\infty f(\alpha, x)dx \right| < \epsilon,$$

which defines the quasi-uniform convergence of the integral $J(\alpha)$ to a limit in (a, b).

As a direct consequence of the above theorem we have our first important result:

THEOREM II. *If all the terms $u_n(\alpha)$ of a series*

$$\Sigma u_n(\alpha) \tag{8}$$

are continuous functions of α in the interval (a, b); and furthermore, if the series

$$\Sigma u_n(\alpha) \frac{x^n}{n!} e^{-x} = e^{-x} u(\alpha, x) \tag{9}$$

converges quasi-uniformly for any positive value of x, then the necessary and sufficient condition that the Borel sum function

$$\overset{\infty}{\underset{0}{S}} u_n(x)$$

be itself continuous, is that the series be quasi-uniformly summable in the interval in question.

(a) Since the terms of (8) are continuous functions of α in (a, b) the terms of (9) are continuous functions of x and α in the region $(0, X, a, b)$, X any finite positive number; and since (9) is quasi-uniformly convergent, its sum is continuous. Furthermore from the definition of quasi-uniform summability of the series

$$J(\alpha) = \overset{\infty}{\underset{0}{S}} u(\alpha) = \int_0^\infty e^{-x} u(\alpha, x)dx$$

is quasi-uniformly convergent in (a, b). Then from the preceding theorem $J(\alpha)$ is continuous, and so the Borel sum function of series (8) is continuous.

* Borel, "Leçons sur les fonctions de variables reélles," p. 9.

Digitized by Google

(*b*) The necessity of the stated conditions follows also from theorem I, where we note that $e^{-x}u(\alpha, x)$ is continuous in $(0, X, a, b)$ and that

$$\int_0^\infty e^{-x}u(\alpha, x)dx$$

is continuous in (a, b). Then this integral converges quasi-uniformly and so the series is quasi-uniformly summable.

The assumption made in the preceding theorem regarding the continuity of

$$\Sigma u_n(\alpha)\frac{x^n}{n!}e^{-x} = e^{-x}u(\alpha, x)$$

may be removed for a very large class of summable series, namely those series the sum of whose first n terms oscillates between finite limits as n increases indefinitely, for in such cases $\Sigma u_n(\alpha)(x^n/n!)e^{-x}$ may be shown to be uniformly convergent in any finite region $(0, X, a, b)$.

To show this we first recall the definition of uniform convergence of a series of functions in two variables. The series $\Sigma f_n(\alpha, x)$, $f_n(\alpha, x)$ being functions of α and x in a region $(0, X, a, b)$ say, is said to be uniformly convergent, to $F(\alpha, x)$, in that region when to an arbitrarily selected ϵ there corresponds a number N such that

$$|F(\alpha, x) - S_n(\alpha, x)| < \epsilon$$

for any $n > N$ and any pair (α, x) in $(0, X, a, b)$.

Dirichlet's test for a uniformly convergent series of functions in one variable is easily extended to functions of two or more variables. It is as follows:

The series $\Sigma a_n f_n(\alpha, x)$ is uniformly convergent in the region $(0, X, a, b)$ provided

1. Σa_n oscillates between finite limits. (The a_n may be functions of α and x in which case $|\Sigma a_n|$ must have its maximum limit less than some fixed number.)

2. For any particular (α, x) in the given interval, $f_n(\alpha, x)$ is positive and never increases with n, and as n tends toward ∞, $f_n(\alpha, x)$ tends uniformly to zero for all values of x and α in the given region.

The proof of this follows exactly that for a series of functions of one variable.*

We proceed now to show that when the series

$$\Sigma f_n(\alpha) \qquad a \leqq \alpha \leqq b \quad (a \text{ and } b \text{ finite})$$

oscillates between finite limits, the series

$$\Sigma f_n(\alpha)\frac{x^n}{n!}e^{-x}$$

* Bromwich, "The Theory of Infinite Series," p. 114.

Digitized by Google

for x in $(0, X)$ (X any finite positive number) satisfies Dirichlet's conditions given above for two variables and hence is uniformly convergent in the region $(0, X, a, b)$.

First, since $\Sigma f_n(\alpha)$ oscillates between finite limits $|\Sigma f_n(\alpha)|$ remains always less than some finite number.

Secondly, the function $f_n(\alpha, x)$ of the definition, which in our case is $(x^n/e^x n!)$, is positive for any value of (α, x) in the region and never increases with n for n sufficiently large, say $n > X$.

Finally we must show that $e^{-x}(x^n/n!)$ tends uniformly to zero for all values of (α, x) in $(0, X, a, b)$.

For α and x fixed, it is well known that

$$\lim_{n\to\infty} \frac{x^n}{e^x n!} = 0.$$

To show that the approach is uniform we proceed as follows: Let $z = (x^n/e^x n!)$ and suppose, for the moment, that n is fixed. Then

$$\frac{dz}{dx} = \frac{nx^{n-1}e^x n! - x^n e^x n!}{e^{2x}(n!)^2}.$$

Setting this expression equal to zero we find the critical values to be $x = 0$ and $x = n$. For $x = 0$ and any n, the function is zero so there is no maximum there; the important case then is when $x = n$. For n some fixed number then, the z function is represented by a curve whose maximum is at the point $x = n$.

For each value of n we obtain one such maximum point and for all n's we obtain a locus of these points defined by the expression

$$\frac{n^n}{e^n n!}, \tag{10}$$

and if we can show that by taking n sufficiently large, greater than some number N, the value of this expression becomes less than ϵ, and remains so for any $n > N$, then our object will have been attained.

By Stirling's formula for n sufficiently large

$$n! \to \sqrt{2\pi n} \cdot \frac{n^n}{e^n}$$

and so, from (10),

$$\frac{n^n}{e^n n!} \to \frac{n^n}{e^n \sqrt{2\pi n} \cdot \dfrac{n^n}{e^n}} = \frac{1}{\sqrt{2\pi n}},$$

i. e., (10) can be made to differ from $1/\sqrt{2\pi n}$ in numerical value by as small a quantity as we choose. But for n sufficiently large $1/\sqrt{2\pi n}$ can be made

Digitized by Google

less than any assigned number ϵ. Consequently for $n > N$, where N is a number such that $|1/\sqrt{2\pi N}| < \epsilon/2$,

$$\left|\frac{n^n}{e^n n!}\right| < \epsilon.$$

We have shown then that the series $\Sigma(f_n(\alpha)x^n/e^n n!)$ satisfies Dirichlet's conditions and is consequently uniformly convergent for any (α, x) in the region $(0, X, a, b)$.

THEOREM III. *The necessary and sufficient condition that the Borel sum function of a series of continuous functions of α in the interval (a, b), $\Sigma u_n(\alpha)$, which oscillates between finite limits, shall be itself continuous in (a, b) is that the series be quasi-uniformly summable (B) in (a, b).*

For according to the considerations immediately above, the series

$$\Sigma u_n(\alpha)\frac{x^n}{n!}e^{-x}$$

is uniformly convergent in the region $(0, X, a, b)$ and hence its sum function is continuous in that region. Consequently since the series is quasi-uniformly summable and hence the Borel integral quasi-uniformly convergent, the conditions of Theorem I are satisfied and the sum function is continuous.

Again if the sum function is continuous the series is obviously quasi-uniformly summable and our theorem is established.

We are in a position now to ascertain to what extent the definition of quasi-uniform summability as given above is consistent with that of quasi-uniform convergence. The discussion of this matter will be limited to series of continuous functions only, since both quasi-uniform summability as well as quasi-uniform convergence are important in so far as they apply to such series.

As a preliminary we establish two theorems concerning quasi-uniform convergent series of continuous functions.

THEOREM IV. *If $\Sigma u_n(\alpha)$ is a quasi-uniformly convergent series of continuous functions in the interval (a, b), a finite number of functions of α, continuous in (a, b), may be added to that series and the resulting series will be quasi-uniformly convergent in (a, b).*

For by Arzela's* theorem the sum function of the given series, say $f(\alpha)$, is continuous in (a, b). If now a finite number of continuous functions is added to $f(\alpha)$, the result will also be a continuous function of α in (a, b). The resulting series, all of whose terms are continuous, has a continuous sum function and consequently is quasi-uniformly convergent.

* Arzela, loc. cit.

Digitized by Google

Now suppose there is given an arbitrarily large positive number λ, and let $[\lambda]$ designate the first integer greater than λ. Consider the set of functions

$$\begin{array}{cccc} \frac{x}{1!}, & \frac{x}{2!}, & \cdots, & \frac{x}{[\lambda]!} \\ \frac{x^2}{1!}, & \frac{x^2}{2!}, & \cdots, & \frac{x^2}{[\lambda]!} \\ \cdot & \cdot & \cdot & \cdot \\ \frac{x^{[\lambda]}}{1!}, & \frac{x^{[\lambda]}}{2!}, & \cdots, & \frac{x^{[\lambda]}}{[\lambda]!} \end{array}$$

all of which are continuous for x in $(0, \lambda)$ and hence bounded there. Since there is a finite number of these functions there will be a finite number M greater than any of them for any x in $(0, \lambda)$.

Let the series of continuous functions of α

$$u_0(\alpha) + u_1(\alpha) + u_2(\alpha) + \cdots + u_n(\alpha) + \cdots \tag{11}$$

be quasi-uniformly convergent for every value of α in (a, b) and form the series of functions

$$u_0(\alpha) + u_1(\alpha) \cdot x + u_2(\alpha) \cdot \frac{x^2}{2!} + \cdots + u_n(\alpha) \frac{x^n}{n!} + \cdots \tag{12}$$

whose terms are continuous functions of x and α in the region $(0, \lambda, a, b)$. We proceed to investigate the series

$$\sum_{\nu=N}^{\infty} u_\nu(\alpha) \frac{x^\nu}{\nu!} \tag{13}$$

where N is any integer greater than $[\lambda]$.

In this series (13) every

$$\frac{x^{N+i}}{(N+i)!} < M,$$

i any positive integer, for every x in $(0, \lambda)$. For, the functions $x^n/n!$ always decrease for $n > x$ and since $N > [\lambda] > \lambda$, λ being the largest value that x may assume, those functions with index N or some larger number will be less than the maximum value of some one of those with index a number from 0 to $[\lambda]$, i. e., they will be less than M for any x in $(0, \lambda)$.

Consequently

$$\left| u_{N+1}(\alpha) \frac{x^{N+1}}{(N+1)!} + \cdots \right| < M \left| u_{N+1}(\alpha) + \cdots \right|. \tag{14}$$

Let us now choose an ϵ arbitrarily small and an integer $N_1 > N$, and set $\epsilon = M \cdot \sigma$, so that by choosing ϵ sufficiently small, M being constant, σ can

Digitized by Google

be made small at pleasure and conversely. Since series (11) is quasi-uniformly convergent, to σ and N_1 there corresponds an integer $N' > N_1$ such that

$$|u_{n_a}(\alpha) + u_{n_a+1}(\alpha) + \cdots| < \sigma \tag{15}$$

where n_a is an integer such that $N' > n_a > N_1$, depending in general upon α in (a, b).

Consequently

$$M|u_{n_a}(\alpha) + \cdots| < M\sigma \qquad N' > n_a > N_1, \quad \alpha \text{ in } (a, b) \tag{16}$$

and since it is independent of x we may say,—for any x in $(0, \lambda)$. (14) and (16) then give

$$\left|u_{n_a}(\alpha)\frac{x^{n_a}}{n_a!} + \cdots\right| < \epsilon \tag{17}$$

for $N' > n_a > N_1$, and for each (x, α) in $(0, \lambda, a, b)$. (17) shows that to ϵ arbitrarily small and N_1 there corresponds a number N', such that for each (x, α) in $(0, \lambda, a, b)$ there exists an integer n_a between N_1 and N' which makes (17) hold. This, however, by our general definition assures the quasi-uniform convergence of (13) with respect to x and α in $(0, \lambda, a, b)$. We have then

THEOREM V. *If a quasi-uniformly convergent series of continuous functions*

$$u_0(\alpha) + u_1(\alpha) + \cdots + u_n(\alpha) + \cdots \qquad \alpha \text{ in } (a, b)$$

be multiplied term by term by the sequence of functions

$$1, x, \frac{x^2}{2!}, \cdots, \frac{x^n}{n!}, \cdots \qquad x \text{ in } (0, \lambda),$$

where λ is some finite number, the resulting series is quasi-uniformly convergent for (x, α) in the region $(0, \lambda, a, b)$.

THEOREM VI. *A quasi-uniformly convergent series of continuous functions is quasi-uniformly summable (B).*

Let the series of continuous functions

$$\Sigma u_n(\alpha) = s(\alpha) \qquad \alpha \text{ in } (a, b) \tag{18}$$

be quasi-uniformly convergent in the interval (a, b). Using (18) form the Borel u-function

$$u(x, \alpha) = u_0(\alpha) + u_1(\alpha) \cdot x + \cdots + u_n(\alpha)\frac{x^n}{n!} + \cdots \tag{19}$$

and the Borel sum

$$\mathop{\mathfrak{S}}_{0}^{\infty} u_n(\alpha) = \int_0^\infty e^{-x}u(x, \alpha)dx = \mathfrak{S}(\alpha). \tag{20}$$

Since (18) is convergent for every α in (a, b) it is also summable (B) with

Digitized by Google

$s(\alpha) = \mathfrak{S}(\alpha)$.* But $s(\alpha)$ is continuous in (a, b) on account of the quasi-uniform convergence of (18), hence $\mathfrak{S}(\alpha)$ is also continuous in (a, b). But

$$\mathfrak{S}(\alpha) = \int_0^\infty e^{-x}u(x, \alpha)dx$$

for every α in (a, b), consequently the Borel integral is a continuous function of α in (a, b). By Theorem V the integrand $e^{-x}u(x, \alpha)$ is continuous in $(0, X, a, b)$, X any finite number, consequently the integral is quasi-uniformly convergent in (a, b) which makes (18) quasi-uniformly summable in that interval and the theorem is established.

An Example.—Consider the series

$$\begin{aligned}(21)\qquad &\alpha - [(\alpha^2 - 1)\alpha + \alpha^3] + 2!\left[\frac{(\alpha^2 - 1)^2\alpha}{2!} + (\alpha^2 - 1)\alpha^3\right] \\ &\qquad - 3!\left[\frac{(\alpha^2 - 1)^3\alpha}{3!} + \frac{(\alpha^2 - 1)^2}{2!}\alpha^3\right] + \cdots \\ &\qquad + (-1)^n n!\left[\frac{(\alpha^2 - 1)^n\alpha}{n!} + \frac{(\alpha^2 - 1)^{n-1}\alpha^3}{(n-1)!}\right] + \cdots.\end{aligned}$$

In the first place this series is convergent for all values of α in $(-1, 1)$. For dividing the $(n + 2)$d term by the $(n + 1)$st we obtain

$$-(n+1)\left[\frac{\dfrac{(\alpha^2 - 1)^{n+1}\alpha}{(n+1)!} + \dfrac{(\alpha^2 - 1)^n}{n!}\alpha^3}{\dfrac{(\alpha^2 - 1)^n\alpha}{n!} + \dfrac{(\alpha^2 - 1)^{n-1}}{(n-1)!}\alpha^3}\right],$$

the limit of which as $n \to \infty$ is $-\alpha^2 + 1$. This is less than 1 for all values of α in the interval except $\alpha = 0$, in which case the series obviously converges to zero. Furthermore since it is convergent it is also summable (B) in the same interval, we proceed to find its Borel sum function $\mathfrak{S}(\alpha)$.

To this end we first form from (21) the Borel u-function using the auxiliary variable x; it is

$$\begin{aligned}u(x, \alpha) &= \alpha - x[(\alpha^2 - 1)\alpha + \alpha^3] + \cdots \\ &\qquad + (-1)^n x^n\left[\frac{(\alpha^2 - 1)^n\alpha}{n!} + \frac{(\alpha^2 - 1)^{n-1}\alpha^3}{(n-1)!}\right] + \cdots \\ (22)\qquad &= \alpha - x(\alpha^2 - 1)\alpha + \cdots + \frac{(-1)^n\alpha x^n(\alpha^2 - 1)^n}{n!} + \cdots \\ &\qquad - x\alpha^3 + x^2\alpha^3(\alpha^2 - 1) - \cdots + \frac{(-1)^n x^n\alpha^3(\alpha^2 - 1)^{n-1}}{(n-1)!} + \cdots,\end{aligned}$$

* G. H. Hardy, *Cambridge Philos. Trans.*, Vol. 19, pp. 298–299.

Digitized by Google

the rearrangement of terms being justified by the result, for each of these lines is in itself an absolutely convergent series, the first line being equal to $\alpha e^{-x(\alpha^2-1)}$, and the second to $-x\alpha^2 e^{-x(\alpha^2-1)}$. Accordingly we have

$$u(x, \alpha) = \alpha e^{-x(\alpha^2-1)} - x\alpha^2 e^{-x(\alpha^2-1)}. \tag{23}$$

Multiplying (23) on both sides by e^{-x} we obtain

$$e^{-x}u(x, \alpha) = \alpha e^{-\alpha^2 x} - x\alpha^2 e^{-\alpha^2 x} = \frac{d(\alpha x e^{-x\alpha^2})}{dx}, \tag{24}$$

since α and x are entirely independent of each other. From (24) we see that the Borel integral,

$$\int_0^\infty \frac{d(\alpha x e^{-x\alpha^2})}{dx}\, dx = \lim_{x\to\infty} (\alpha x e^{-x\alpha^2}) = 0, \tag{25}$$

gives $\mathfrak{S}(\alpha) = 0$ for every α in $(-1, 1)$, it is then continuous, and since the terms of (21) are continuous functions of α in $(-1, 1)$ the limit is approached at least quasi-uniformly. It is not uniform convergence, consequently it is just quasi-uniform convergence. The series (21) is then quasi-uniformly summable.

Finally, since the sum of (21) is equal to the limit in (25) for each value of α in $(-1, 1)$ the sum function of (21) is continuous and consequently (21) is quasi-uniformly convergent in $(-1, 1)$.

We conclude then from the above investigation that a quasi-uniformly convergent series will not in general be more than quasi-uniformly summable when summed according to Borel's method, i. e., that the limit used in Borel's method of summation is not approached any "more uniformly" than is the limit used in ordinary summation of a convergent series.

THEOREM VII. (I) *Let the series* $\Sigma u_n(\alpha)$ *be summable* (B) *for every* α *in* (a, b), *and uniformly summable except at a finite set of points* $\alpha_1, \alpha_2, \cdots, \alpha_s$ *of* (a, b). *Furthermore* (II), *let*

$$\lim_{\alpha\to\alpha_i} \int_{n_i}^\infty e^{-x}u(x, \alpha)dx = 0,$$

for n_i *a finite integer greater than* v_i, *where the latter is arbitrarily selected. Then* $\Sigma u_n(\alpha)$ *is quasi-uniformly summable in* (a, b).

From (II) to an ϵ and an N arbitrarily chosen, for each α_i there exists a $\delta_i > 0$ such that

$$\left| \int_{n_i}^\infty e^{-x}u(x, \alpha)dx \right| < \epsilon,$$

when n_i is some finite integer greater than N and α is in the partial interval $(\alpha_i - \delta_i, \alpha_i + \delta_i)$. Then for all the α_i there will be a set of numbers $n_1, n_2, \cdots n_s$ all greater than N. Since there is but a finite number there will be one

Digitized by Google

greater than all the others; let N' be an integer greater than this, then choosing δ the smallest of the δ_i we have

$$\left|\int_{n_i}^{\infty} e^{-x}u(x, \alpha)dx\right| < \epsilon \quad N' > n_i > N \text{ and } \alpha \text{ in } (\alpha_i - \delta, \alpha_i + \delta). \tag{26}$$

From (a, b) delete the subintervals of width 2δ at each point α_i and call that which remains B_δ. By (I) $\Sigma u_n(\alpha)$ is uniformly summable in B_δ, we have then corresponding to ϵ an N'' such that

$$\left|\int_{n}^{\infty} e^{-x}u(x, \alpha)dx\right| < \epsilon \quad \text{for any } n \geq N'' \text{ and for any } \alpha \text{ in } B_\delta. \tag{27}$$

From (26) and (27), by choosing the larger of N' and N'' and calling it N_1, we have—to ϵ and N there corresponds a finite number N_1 such that

$$\left|\int_{n_\alpha}^{\infty} e^{-x}u(x, \alpha)dx\right| < \epsilon \qquad \text{for } N_1 > n_\alpha > N,$$

where n_α is one of a finite number of numbers depending, in general, on α, a point of (a, b). This, however, is the definition for the quasi-uniform convergence of the integral in (II), and hence for the quasi-uniform summability of the series in (I).

Corollary.—The series of continuous functions $\Sigma u_n(\alpha)$ summable (B) for every α in (a, b), and uniformly summable except at a finite set of points $\alpha_1, \alpha_2, \cdots \alpha_s$ of (a, b), and for which points

$$\lim_{\alpha \to \alpha_i} \int_{n_i}^{\infty} e^{-x}u(x, \alpha)dx = 0$$

where n_i is a finite integer greater than an arbitrarily large finite number v_i, such a series has a continuous Borel sum function in (a, b).

§ 4. CESÀRO'S METHOD OF SUMMATION

Cesàro defined the sum of a series Σu_n to be

$$\lim_{n \to \infty} \frac{s_0 + s_1 + \cdots + s_n}{n + 1}$$

where $s_i = u_0 + u_1 + \cdots + u_i$; or more generally the sum of Σu_n is

$$\lim_{n \to \infty} \frac{S_n^{(r)}}{A_n^{(r)}}, \tag{28}$$

wherein

$$S_n^{(r)} = s_n + rs_{n-1} + \cdots + \frac{r(r+1)\cdots(r+n-1)}{n!}s_0$$

Digitized by Google

and

$$A_n^{(r)} = \frac{(r+1)(r+2)\cdots(r+n)}{n!},$$

provided, in both cases, that the limit existed.

If the u_n are functions of α in the interval (a, b) then (28) would read

$$\lim_{n\to\infty} \frac{S_n^{(r)}(\alpha)}{A_n^{(r)}} = S(\alpha). \tag{29}$$

Definition.—The series $\Sigma u_n(\alpha)$ is said to be quasi-uniformly summable (C) for α in the interval (a, b) if the limit

$$\lim_{n\to\infty} \frac{S_n^{(r)}(\alpha)}{A_n^{(r)}} = S(\alpha)$$

is approached quasi-uniformly in (a, b).

THEOREM VIII. *The necessary and sufficient condition that the series of continuous functions $\Sigma u_n(\alpha)$, for α in (a, b), have a continuous sum in the Cesàro sense is that it be quasi-uniformly summable (C) in (a, b).*

Proof.—Since $u_0(\alpha)$, $u_1(\alpha)$, $\cdots$ are continuous functions of α in (a, b), the functions $s_n(\alpha) = u_0(\alpha) + u_1(\alpha) + \cdots + u_n(\alpha)$, n being some finite integer, are also continuous functions of α in (a, b), and therefore the series of which

$$\frac{S_n^{(r)}(\alpha)}{A_n^{(r)}}$$

is the sum of the first $n + 1$ terms is a series of continuous functions of α in (a, b). This last series, however, according to our hypothesis converges quasi-uniformly in (a, b), and consequently its sum function is continuous. But the sum function of this series is precisely the Cesàro sum of $\Sigma u_n(\alpha)$, hence the first part of the theorem holds.

On the other hand, when $S(\alpha)$, as well as the $u_n(\alpha)$, is continuous, which is again the case with

$$\frac{S_n^{(r)}(\alpha)}{A_n^{(r)}},$$

the limit (29) is approached quasi-uniformly, and then by the definition the series $\Sigma u_n(\alpha)$ is quasi-uniformly summable.

THEOREM IX. *Every quasi-uniformly convergent series of continuous functions is quasi-uniformly summable (C).*

Let the series $\Sigma u_n(\alpha)$, whose terms are continuous functions of α in (a, b), be itself quasi-uniformly convergent in that interval. Form the Cesàro sum

$$\lim_{n\to\infty} \frac{S_n^{(r)}(\alpha)}{A_n^{(r)}} = S(\alpha).$$

Digitized by Google

Chapman has proved that every convergent series is summable (C_r) with sum equal to the ordinary sum.* Hence for each α in (a, b) the ordinary sum $s(\alpha) = S(\alpha)$, and the same reasoning as on p. 14 may be pursued, giving the theorem at hand.

THEOREM X. *The series $\Sigma u_n(\alpha)$ summable (C_r) for every α in (a, b) and uniformly summable (C_r) for α in (a, b) except at a finite set of points $\alpha = \alpha_1, \alpha_2, \cdots \alpha_s$; and, further, for each α_i*

$$\lim_{\alpha \to \alpha_i} \left(S(\alpha) - \frac{S_{n_i}^{(r)}(\alpha)}{A_{n_i}^{(r)}} \right) = 0,$$

where n_i is a finite integer greater than an arbitrarily chosen finite v_i, is quasi-uniformly summable (C_r) in (a, b).

The proof of this follows directly that of p. 16.

Corollary.—The series of continuous functions $\Sigma u_n(\alpha)$ summable (C_r) for every α in (a, b) and uniformly summable (C_r) except at a finite set of points $\alpha_1, \alpha_2, \cdots, \alpha_s$ of (a, b), and such that

$$\lim_{\alpha \to \alpha_i} \left(S(\alpha) - \frac{S_{n_i}^{(r)}(\alpha)}{A_{n_i}^{(r)}} \right) = 0$$

for n_i a finite integer greater than an arbitrarily selected finite v_i, has a continuous Cesàro sum function in (a, b).

§ 5. THE "CESÀRO-RIESZ" SUMS OF HARDY AND CHAPMAN

Before proceeding to the "Cesàro-Riesz" sums of Hardy and Chapman we establish some general theorems concerning m-tuple limits.

THEOREM XI. *Let $S_{p_1, p_2, \cdots, p_m}(x_1, x_2, \cdots, x_n)$ be a function of n variables x_i and m parameters p_i. Let*

$$\lim_{\substack{p_1 \to \infty \\ p_2 \to \infty \\ \vdots \\ p_m \to \infty}} S_{p_1 p_2 \cdots p_m}(x_1, x_2, \cdots, x_n) = s(x_1, x_2, \cdots, x_n)$$

for each x in the n-way space X. Let $S_{p_1, p_2, \cdots, p_m}(x_1, \cdots x_n)$ be continuous functions of $x_1, x_2, \cdots, x_n$, in X for each fixed set of values of $p_1, p_2, \cdots, p_m$. Then the necessary and sufficient condition that $s(x_1, x_2, \cdots, x_n)$ be a continuous function of $x_1 \cdots x_n$ in X is that the limit,

$$\lim_{\substack{p_1 \to \infty \\ \vdots \\ p_m \quad \infty}} S_{p_1 p_2 \cdots p_m}(x_1, x_2, \cdots, x_n) = s(x_1, x_2, \cdots, x_n)$$

be approached quasi-uniformly in X.

* Chapman, *Proc. Lond. Math. Soc.*, Ser. 2, Vol. IX, 1911, p. 377.

Digitized by Google

The argument used to prove this is precisely that used in Theorem I, so we choose to omit it here.

As a special case of Theorem XI we have

THEOREM XII. *If the Pringsheim double limit $S_{p,q}(\alpha)$ exists for every α in (a, b), and if for each pair of values of p and q, $S_{p,q}(\alpha)$ is continuous for α in (a, b), then the necessary and sufficient condition that the "sum function,"*

$$\lim_{p,\, q\to\infty} S_{p,q}(\alpha) = s(\alpha),$$

be continuous in (a, b) is that this limit be approached quasi-uniformly.

We proceed now to consider the Cesàro-Riesz sums. From the series Σu_ν we form the auxiliary series

$$\lim_{p,\, n\to\infty} \sum_{\nu=0}^{[n]} u_\nu \left\{1 - \frac{\lambda(\nu)}{\lambda(n)}\right\}^{\lambda(n)/p},$$

where n may take any real positive values whatever, $[n]$ represents the first integer greater than n and $\lambda(n)$ is any function of n which increases to infinity simultaneously with n. If the limit,

$$\lim_{p,\, n\to\infty} \sum_{\nu=0}^{[n]} u_\nu \left\{1 - \frac{\lambda(\nu)}{\lambda(n)}\right\}^{\lambda(n)/p} = \lim_{p,\, n\to\infty} S(n, p), \tag{30}$$

say, exists as p and n approach infinity simultaneously but independently, the series is said to be summable.

It may happen, however, that the limit (30) does not exist but that

$$\lim_{\substack{p,\, n\to\infty \\ F}} \sum_{\nu=0}^{[n]} u_\nu \left\{1 - \frac{\lambda(\nu)}{\lambda(n)}\right\}^{\lambda(n)/p} \tag{31}$$

does exist, where the F designates some path over which n and p pass in going to infinity, i. e.,

$$\lambda(n) = F(p).$$

In this case we say the series is summable (C, R, F).

Definition.—Let the series of functions $\Sigma u_\nu(\alpha)$, of α in (a, b), be summable by the Cesàro-Riesz sum for every α in (a, b). If the limit,

$$\lim_{p,\, n\to\infty} S(n, p, \alpha) = \mathbf{S}(\alpha),$$

is approached quasi-uniformly in (a, b), we shall say the series is quasi-uniformly summable (C, R) in (a, b).

In particular, if

$$\lim_{\substack{p,\, n\to\infty \\ F}} S(n, p, \alpha) = \mathbf{S}(\alpha)$$

Digitized by Google

quasi-uniformly in (a, b), where the path F is $\lambda(n) = d \cdot p$, where d is some constant; or F: $\lambda(n) = p\omega(n)$; we shall say that $\Sigma u_\nu(\alpha)$ is quasi-uniformly summable (R, λ, d), or (R, λ, ω) respectively, in (a, b).

Let

$$\Sigma u_\nu(\alpha) \tag{32}$$

be a series of continuous functions of α in (a, b). Form the series

$$\lim_{p, n \to \infty} \sum_{\nu=0}^{[n]} u_\nu(\alpha) \left\{ 1 - \frac{\lambda(\nu)}{\lambda(n)} \right\}^{\lambda(n)/p} = \lim_{p, n \to \infty} S(n, p, \alpha) = \mathbf{S}(\alpha), \tag{33}$$

wherein $\{1 - \lambda(\nu)/\lambda(n)\}^{\lambda(n)/p} = f_\nu(n, p)$, say, is a function defined for all positive values of n and p, and for $\nu = 1, 2, \cdots, [n]$.

THEOREM XIII. *Given the series* (32) *for which the limit in* (33) *exists for every α in (a, b), i. e.,* (32) *is summable (C, R) for every α in (a, b). The necessary and sufficient condition that* (32) *have a continuous Cesàro-Riesz sum is that it be quasi-uniformly summable (C, R) in (a, b).*

Since (32) is a series of continuous functions of α in (a, b) for a fixed n and p,

$$S(n, p, \alpha) = \sum_{\nu=0}^{[n]} u_\nu(\alpha) \left\{ 1 - \frac{\lambda(\nu)}{\lambda(n)} \right\}^{\lambda(n)/p},$$

is also continuous with respect to α in (a, b). Furthermore, since the limit (33) is approached quasi-uniformly, due to the quasi-uniform summability of $\Sigma u_\nu(\alpha)$, $\mathbf{S}(\alpha)$ is continuous by Theorem XI. The second part follows as in the former cases. As a direct consequence of this we have the

Corollary.—A series of continuous functions $\Sigma u_\nu(\alpha)$ summable (R, λ, d) or (R, λ, ω) for every α in (a, b), has a continuous sum function if and only if it is quasi-uniformly summable (R, λ, d) or (R, λ, ω) respectively in (a, b).

For in these cases we have

$$\lim_{\substack{p, n \to \infty \\ F}} S(n, p, \alpha) = \lim_{\substack{p, n \to \infty \\ F}} \sum_{\nu=0}^{[n]} u_\nu(\alpha) \left\{ 1 - \frac{\lambda(\nu)}{\lambda(n)} \right\}^{\lambda(n)/p} = \mathbf{S}(\alpha),$$

where $\overset{p, n \to \infty}{F}$ designates, as above, that n and p approach infinity simultaneously but along a path F. In the case of (R, λ, d), F: $\lambda(n) = dp$, where d is a constant, and then we obtain

$$S(n, p, \alpha) = \sum_{\nu=0}^{[n]} u_\nu(\alpha) \left\{ 1 - \frac{\lambda(\nu)}{dp} \right\}^{d},$$

which is obviously a special case of the above.

Likewise in the case where we have summability (R, λ, ω), F: $\lambda(n) = \omega(n)p$, we obtain another special case.

Digitized by Google

We have still to consider the case of repeated limits, i. e., the case where first one of the parameters tends to infinity and then after this, the second tends to infinity. For this case we have the following general theorem:

THEOREM XIV. *Let*

(*a*) $\lim_{p\to\infty}\lim_{n\to\infty} S(n, p, \alpha) = \mathbf{S}(\alpha)$ *for every* α *in* (a, b);

(*b*) $S(n, p, \alpha)$ *be a continuous function of* α *for every fixed* n *and* p;

(*c*) $\lim_{n\to\infty} S(n, p, \alpha) = \mathbf{S}(p, \alpha)$ *exist for every positive value of* p, *and every* α *in* (a, b).

Then the necessary and sufficient conditions that $\mathbf{S}(p, \alpha)$ *and* $\mathbf{S}(\alpha)$ *be continuous functions of* α *in* (a, b) *are that the limits,*

$$\lim_{n\to\infty} S(n, p, \alpha) = \mathbf{S}(p, \alpha) \qquad \textit{and} \qquad \lim_{p\to\infty} \mathbf{S}(p, \alpha) = \mathbf{S}(\alpha)$$

are both approached quasi-uniformly, in the first case for p *fixed and* α *in* (a, b), *in the second case for* α *in* (a, b).

This theorem is proved in a manner precisely analogous to those already demonstrated, and also without mention of the fact that n and p must or must not be integers; we assume them always positive, however.

In applying this theorem to the case of summable series we proceed after the following manner.

Consider the function

$$S(n, p, \alpha) = \sum_{\nu=0}^{[n]} u_\nu(\alpha) \left\{ 1 - \frac{\lambda(\nu)}{\lambda(n)} \right\}^{\lambda(n)/p} \tag{34}$$

defined, as above, for all values of n and p in the interval $(0, \infty)$ and all α's in (a, b).

As before, let $\Sigma u_\nu(\alpha)$ be a series of continuous functions of α. The terms of (34) are then all continuous with respect to α and so $S(n, p, \alpha)$ is a continuous function of α for each n and p fixed.

Let the limit of (34) exist for each p fixed and $n \to \infty$, and let $\Sigma u_\nu(\alpha)$ be summable by repeated limits for every α in (a, b). Then we have precisely conditions (*a*), (*b*) and (*c*) of the preceding theorem fulfilled, so we may apply that theorem to summable series and obtain the following theorem.

THEOREM XV. *Given a series of continuous functions* $\Sigma u_\nu(\alpha)$ *summable by repeated limits for every* α *in* (a, b). *Then if the limit,* $\lim_{n\to\infty} S(n, p, \alpha) = \mathbf{S}(p, \alpha)$, *exists for each* p *in* $(0, \infty)$ *and every* α *in* (a, b), *the necessary and sufficient conditions that the* $\mathbf{S}(p, \alpha)$ *and* $\mathbf{S}(\alpha)$, *the sum of the series, be continuous functions of* α *in* (a, b) *are that the limits*

$$\lim_{n\to\infty} S(n, p, \alpha) = \mathbf{S}(p, \alpha), \qquad \lim_{p\to\infty} \mathbf{S}(p, \alpha) = \mathbf{S}(\alpha)$$

be approached quasi-uniformly for α *in* (a, b).

Digitized by Google